

Exploring loci of points by DGS and CAS in teaching geometry

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Abstract

The article is aimed at searching for loci of points of given properties using dynamic geometry systems (DGS) and computer algebra systems (CAS) in teaching geometry. Loci of points belong to difficult topics of school curricula at all levels of mathematics education. New technologies can facilitate these topics. By DGS we demonstrate a problem. DGS also enables us to find the searched locus and state a conjecture. To identify the locus equation we use CAS which perform symbolic computations. In addition, in those cases, where it is possible, a classical proof is given as well. This method is demonstrated in several examples from 2D and 3D geometry.

1 Introduction

Loci of points belong to difficult topics of school curricula at all levels of mathematics education, although they have significant importance of motivation and discovery. The development of computers and mathematical software allows such activities that previously were not possible. There are many interesting loci — besides well known conics also algebraic curves of higher degree such as cubics, quartics etc. — which students can recognize. It turns out that such curves and their properties could become a common part of a school curricula in the near future. Moreover new technologies and mathematical software facilitate investigating loci in a 3D space as well. From this point of view students could encounter cubic surfaces and surfaces of higher degree as important loci in their textbooks soon.

The paper presents our experience with exploring of loci of points of given properties with computer support at the university course on geometry. This experience could also serve as a method for searching for loci in teaching geometry.

We will show the technique we are using at a university geometry seminar on using computer methods to solve problems in elementary geometry. One of the authors leads this geometry seminar for several years. The students who take part in this seminar are mostly participants of teacher's training of mathematics in their 4th year study, i.e. they have knowledge at the level of a basic course in

geometry. The seminar is aimed at various proving techniques including automated geometry theorem proving. In the seminar classical proving with paper and pencil is often compared with computer proving [12]. After the introduction of basic knowledge from the theory of automated geometry theorem proving students work on their seminar works with topics often taken from [1]. Seminar works consist of the following parts: Introduction of a problem, Demonstration in DGS, Visual evidence in DGS (visual proof), Classical proof, Automated (computer) proof (in CAS). Note that under automated (computer) proof we mean interactive proof, i.e. performing several steps (Introduction of a coordinate system, Translation of geometric properties into the set of algebraic equations and inequations, Computation of the normal form of a conclusion polynomial with respect to the Gröbner basis of the hypotheses ideal, ...) which are necessary to prove a statement, see [12] for details. From this point of view the way applied in [11] seems to be promising in interactive proving in teaching geometry.

Students mostly prefer classical proving to computer proving as it enables real insight into the problem. But in the case they do not have a "key idea" to prove the problem classically, they think about computer proving. As computer proving (we mean its interactive form) is quite new for students, let us give some remarks on our experience with computer proving in a geometric seminar, see also [13]:

- Students learn that there exists "modern" powerful method to solve problems.
- Classical and computer approaches are complementary.
- Students have many problems with computer solving — more than expected.
- Students rely more on the computer instead of themselves.
- Problems with translation from algebra to geometry, e.g. to explain degeneracy conditions geometrically, students prefer an algebraic formulation to a geometric one, etc.
- Automatic geometry theorem proving requires deep geometric knowledge.
- There are problems which we are not able to solve classically, e.g. most non-Euclidean constructions can be carried out by computer.
- There are problems which we are not able to solve even by computer — e.g. in connection with completeness of the method.
- The main principle - switching between geometry and algebra.
- Motivation for geometry, algebra and learning English.

The paper is structured as follows: Section 2 gives basic knowledge of exploring loci of points, in Section 3 loci in a plane are investigated in two examples and finally in Section 4 we demonstrate a locus in 3D.

2 Exploring loci of points

Derivation of geometry theorems belongs to mighty tools of automated geometry theorem proving. By elimination of suitable variables in the system of algebraic equations, describing geometric properties of given objects, we get the required formulas. From this point of view searching for loci equations can be considered as derivation (discovering) of algebraic formulas which the locus point (tracer) must satisfy while the mover point moves along its path [14], [3].

By exploring loci we first use DGS to demonstrate a problem. DGS programs GeoGebra¹ [6] and Cabri Geometry II [8] are often used at the geometry course. Students can construct geometric figures and dynamically change them by moving points. We can create a hypothesis by using the DGS for creating a trace for the point.

There exist many DGS programs with some minor differences which behave in a similar way when obtaining loci. In general, two objects must be selected: the first one, usually a point called *driving point* or *mover*, is bound to a path, whereas the other, the *locus point* or *tracer*, depends somehow on the first one. Since the element dependence is preserved while a driving point traverses its path, the locus is the trajectory of the locus point [2].

Next important part comes after verification by visual evidence in DGS (visual proof) — searching for locus equation or its characteristic property. Usually we need to choose suitable coordinate system and decide, where to place points and objects to obtain the simplest expressions.

Then we translate geometric properties of the problem into algebraic equations and inequations. By CAS we get the locus equation from the system of these equations and inequations using elimination of variables. Elimination is implemented in most CAS. In the paper we will use the program CoCoA², [4] which is based on Gröbner bases computation [5], and the program Epsilon³, [18], [17] which is based on Wu–Ritt characteristic sets approach [19]. In most cases by elimination of coordinates of the mover we obtain the locus equation of our searched curve/surface.

3 Loci in plane

This section is concerned with searching for loci of points in plane. We will give two examples of plane loci.

3.1 Asteroid

This example shows one property of an asteroid [15] (Fig. 3). Let us solve the following problem:

Let k be a circle with two orthogonal diameters x, y . Denote by A and B the feet of perpendiculars to x and y from an arbitrary point $C \in k$. Find the locus of M which is on AB such that CM is perpendicular to AB when C moves along k .

We can define:

¹Software GeoGebra is freely distributed at the address <http://www.geogebra.at>

²Software CoCoA is freely distributed at <http://cocoa.dima.unige.it>

³Software Epsilon is freely distributed at <http://www-calfor.lip6.fr/Xwang/epsilon/>

Definition 1 *The locus above we call "A-perpendicular".*

First we demonstrate the locus in GeoGebra. Using the button Trace on (Fig. 1) or the button Locus

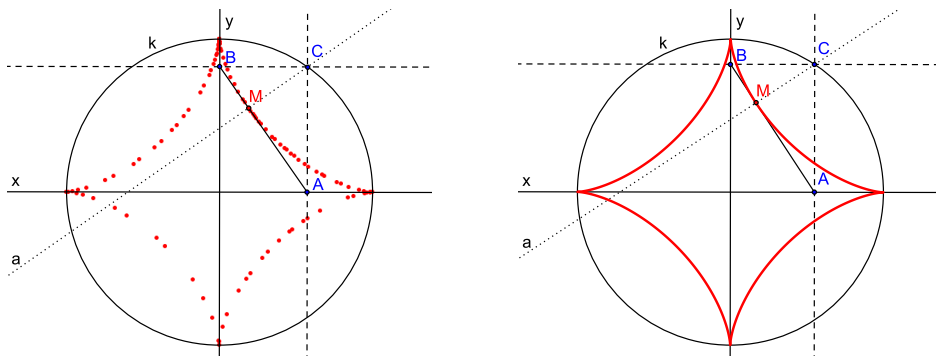


Figure 1: Locus by "Trace on" button Figure 2: Locus by "Locus" button

(Fig. 2) we construct the locus of M as C moves along k .

The visual appearance produced by GeoGebra indicates an asteroide. In order to verify this assumption we search for the equation of this curve.

For that purpose we place a coordinate system so that $A = [p, 0]$, $B = [0, q]$, $C = [p, q]$, $M = [m, n]$ and let k be a circle centered at $O = [0, 0]$ with radius a , $a \neq 0$ (Fig. 3)¹.

Translation of geometry properties of objects into the set of polynomial equations gives

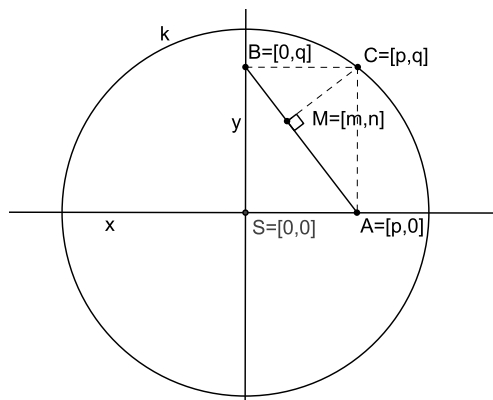


Figure 3: Exploring the locus of M with moving C

$$\begin{aligned} M \in AB &\Leftrightarrow H_1 := qm + pn - pq = 0, \\ CM \perp AB &\Leftrightarrow H_2 := pm - qn - p^2 + q^2 = 0, \\ C \in k &\Leftrightarrow H_3 := p^2 + q^2 - a^2 = 0. \end{aligned}$$

We get the system of three equations $H_1 = 0$, $H_2 = 0$, $H_3 = 0$ in variables p, q, m, n, a . To find the locus of $M = [m, n]$ we eliminate variables p, q in the ideal $I = (H_1, H_2, H_3)$ and get a relation in m, n which depends on a .

¹Instead of a we can choose the value 1 to simplify the problem. But we will keep the value a to obtain the locus equation depending on this parameter as we can see in [9], [15]

In CoCoA we enter

```
Use R:=Q[p,q,m,n,a];
I:=Ideal(qm+pn-pq,pm-qn-p^2+q^2,p^2+q^2-a^2);
Elim(p..q,I);
```

and get the elimination ideal

```
Ideal(-2/9m^6a^2-2/3m^4n^2a^2-2/3m^2n^4a^2-2/9n^6a^2+2/3m^4a^4-
-14/3m^2n^2a^4+2/3n^4a^4-2/3m^2a^6-2/3n^2a^6+2/9a^8)
```

which is generated by the only polynomial which, after factoring, with respect to $a \neq 0$, leads to the equation

$$m^6 + 3m^4n^2 + 3m^2n^4 + n^6 - 3m^4a^2 + 21m^2n^2a^2 - 3n^4a^2 + 3m^2a^4 + 3n^2a^4 - a^6 = 0. \quad (1)$$

Relation (1) can be written as

$$(m^2 + n^2 - a^2)^3 + 27m^2n^2a^2 = 0$$

which is the equation of an asteroid [9],[15].

Another form of an asteroid is

$$n^2 - (a^{2/3} - m^{2/3})^3 = 0.$$

Transforming the equation into an explicit form we get two equations

$$n_1 = \sqrt{(a^{2/3} - m^{2/3})^3} \quad \text{and} \quad n_2 = -\sqrt{(a^{2/3} - m^{2/3})^3}.$$

In the following figures we display the locus as a graph of two functions (Fig. 4) or like an implicit curve (Fig. 5)

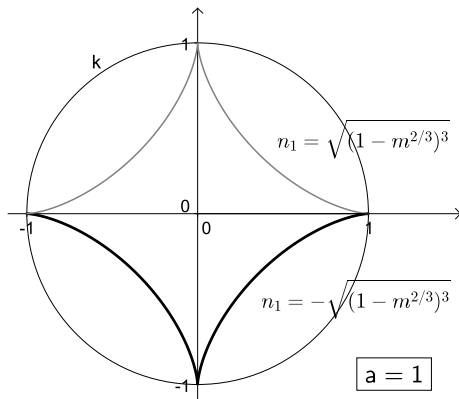


Figure 4: Asteroid as two functions

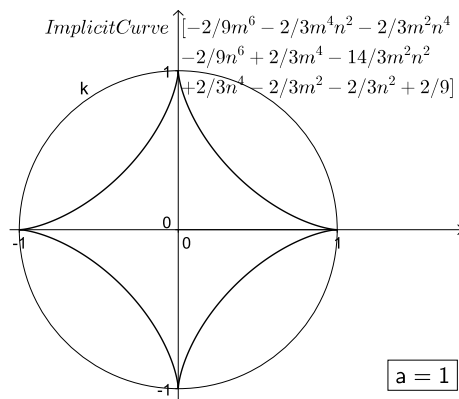


Figure 5: Asteroid as implicit curve

curve using the command ImplicitCurve in GeoGebra (Fig. 5).

Let us investigate the locus which is given in the following [16]:

Consider a circle l with radius $a/4$ rolling about the inside of a fixed circle k with radius a . Find the locus of a point M on l .

Let us define:

Definition 2 *The locus above we call "A-rolling".*

In the Fig. 6 we can see the locus which is created in the way above.

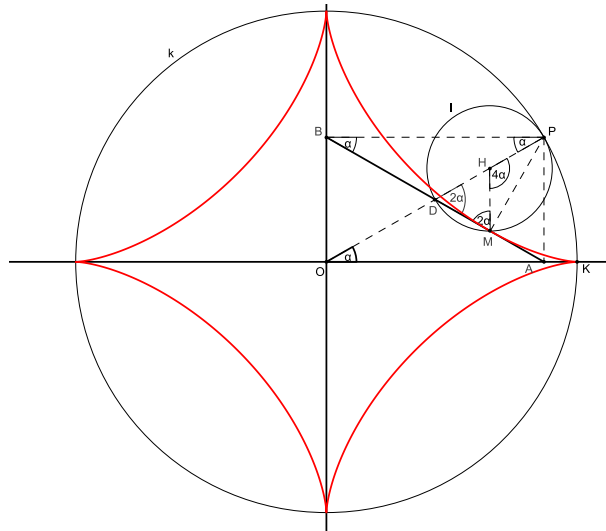


Figure 6: Locus created by circle rolling inside

The loci defined by Definitions 1 and 2 are closely connected as we can see in the next theorem:

Theorem 3 *Both loci — "A-perpendicular" and "A-rolling" from Definitions 1 and 2 are equivalent.*

We can prove the theorem both classically and by computer support. Computer approach is similar to that we used above. Let us demonstrate a classical proof which is very important in teaching geometry and if feasible we prefer it. It is as follows:

Proof.

Realize that when a small circle l with radius $a/4$ rolls about the inside of a fixed circle k with radius a that the lengths of both arcs PK and PM of circles k and l are the same. This means that if $|\angle KOP| = \alpha$ then $|\angle MHP| = 4\alpha$, Fig. 6. Further the triangle DMH is isosceles with $\angle D = \angle M = 2\alpha$. This implies that M lies on AB and by the theorem of Thales applied to a triangle PMD the relation $PM \perp AB$ follows. Similarly we can prove the opposite implication. ■

3.2 Strophoid

In the following problem we will explore one property of a strophoid [9], [15].

Let ABC be a triangle with the given side AB and the vertex C on a circle k centered at A and radius $|AB|$. Find the locus of the orthocenter M of ABC as C moves along k .

First we describe the problem in GeoGebra (Fig. 7). Using the button Trace on (Fig. 8) or the button Locus (Fig. 9) in GeoGebra we construct the locus of M when C moves along k .

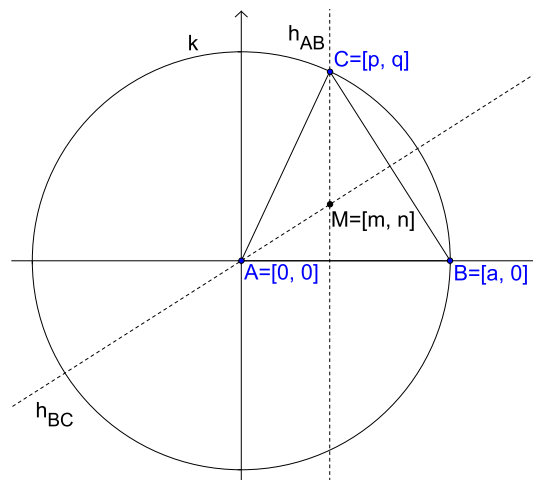


Figure 7: Exploring the locus of orthocenter M with moving C

Now we will derive the locus equation. For that purpose we choose a rectangular coordinate system such that $A = [0, 0]$, $B = [a, 0]$, $C = [p, q]$ and $M = [m, n]$, where $a \neq 0$ (Fig. 7). Then

$$\begin{aligned} M \in h_{AB} &\Leftrightarrow H_1 := m - p = 0, \\ M \in h_{BC} &\Leftrightarrow H_2 := (p - a)m + qn = 0, \\ C \in k &\Leftrightarrow H_3 := p^2 + q^2 - a^2 = 0. \end{aligned}$$

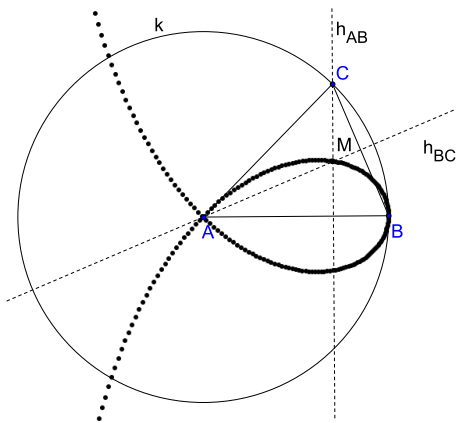


Figure 8: Locus by "Trace on" button

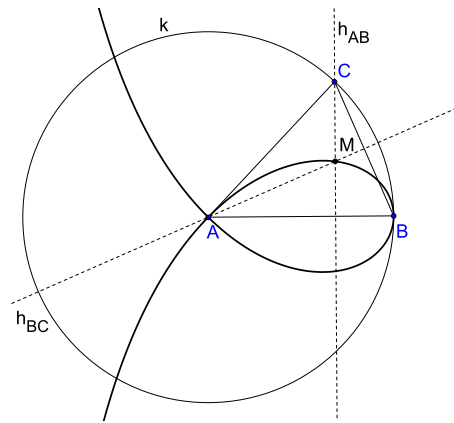


Figure 9: Locus by "Locus" button

Elimination of p, q in the system $H_1 = 0, H_2 = 0, H_3 = 0$ in the program Epsilon gives

```
with(epsilon);
U:= [m-p, (p-a)*m+q*n, p^2+q^2-a^2]:
X:= [m, n, p, q]:
CharSet(U, X);
```

the answer

$$[-a \cdot n^2 + m^2 \cdot a - m^3 - m \cdot n^2, m-p, m \cdot p - m \cdot a + q \cdot n],$$

$$\text{'factors removed'} = \{-m+a\}$$

and we get the only equation

$$-an^2 + m^2a - m^3 - mn^2 = 0 \tag{2}$$

which is a cubic curve called strophoid [9], [15].

Another form of (2) is

$$n^2(a + m) - m^2(a - m) = 0$$

which can be expressed as the union of two functions

$$n = \pm m \cdot \sqrt{\frac{a - m}{a + m}},$$

(Fig. 10) or like implicitplot (Fig. 11).

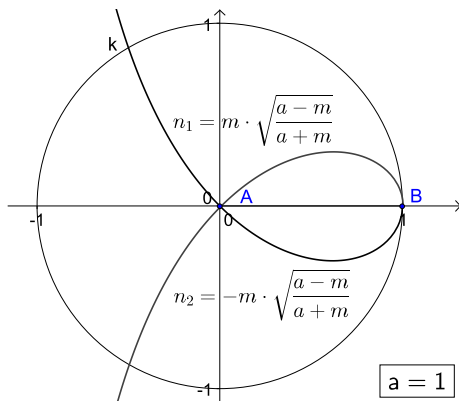


Figure 10: Strophoid as two functions

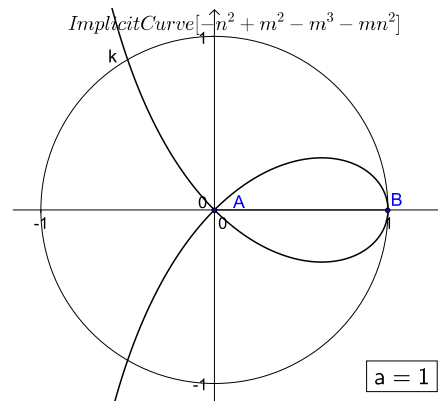


Figure 11: Strophoid as implicit curve

4 Loci in 3D

The next example will be similar to the previous one with the only change — we try to construct the locus in 3D space.

We will solve the following problem:

Let ABC be a triangle with the given side AB and the vertex C on a sphere κ centered at A and radius |AB|. Find the locus of the orthocenter M of ΔABC when C moves on κ.

For illustration of this example we use 3D window of GeoGebra 5 [7] (Fig. 12). Plane ρ₁ through C is perpendicular to AB, plane ρ₂ is perpendicular to BC and plane of ΔABC is the third, where M is located.

In order to get simple equations we place a coordinate system so that A = [0, 0, 0], B = [a, 0, 0], C = [p, q, r], M = [m, n, o] and let κ be a sphere centered at A with radius a, a ≠ 0 (Fig. 13).

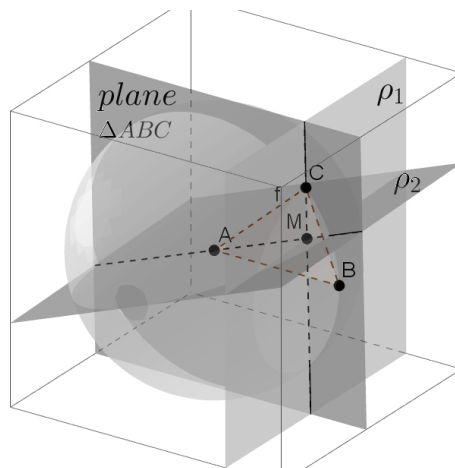


Figure 12: View in GeoGebra 3D

Translation of geometric statements into the following set of polynomial equations gives:

$$\begin{aligned}
 M \in \rho_1 &\Leftrightarrow H_1 := m - p = 0, \\
 M \in \rho_2 &\Leftrightarrow H_2 := pm + qn + ro - pa = 0, \\
 M \in ABC &\Leftrightarrow H_3 := -arn + aqo = 0, \\
 C \in k &\Leftrightarrow H_4 := p^2 + q^2 + r^2 - a^2 = 0.
 \end{aligned}$$

We get the system of four equations $H_1 = 0, H_2 = 0, H_3 = 0$ and $H_4 = 0$ in variables p, q, r, m, n, o, a .

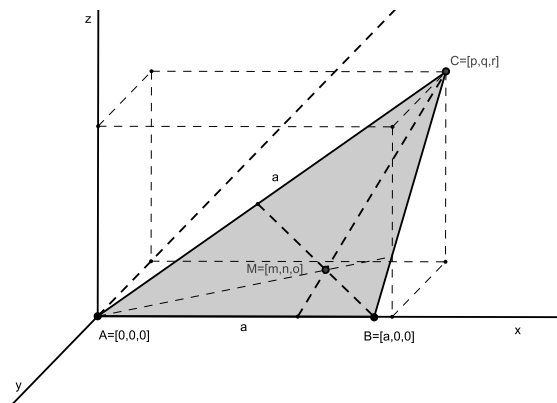


Figure 13: Searching for locus equation

To find the locus of $M = [m, n, o]$ we eliminate variables p, q, r in the ideal $I = (H_1, H_2, H_3, H_4)$ to get a relation in m, n, o which depends on a . We enter in CoCoA

```

Use R:=Q[p, q, r, m, n, o, a];
I:=Ideal(m-p, pm+qn+ro-pa, -arn+aqo, p^2+q^2+r^2-a^2);
Elim(p..r, I);
    
```

and get

$$\text{Ideal}(m^4a + m^2n^2a + m^2o^2a - 2m^3a^2 + m^2a^3 - n^2a^3 - o^2a^3).$$

By factoring and rewriting this polynomial as an equation we get

$$a(-m + a)(am^2 - an^2 - ao^2 - m^3 - mn^2 - mo^2) = 0. \quad (3)$$

By elimination extraneous factors can occur in the locus equation. We can see them in (3). The case $a = 0$ can be ruled out as $a \neq 0$. The factor $-m + a$ can also be ruled out since if $m = a$ then by assumptions $M = B = C$ and $M = [a, 0, 0]$ is already involved in (4). Hence the locus equation is

$$am^2 - an^2 - ao^2 - m^3 - mn^2 - mo^2 = 0$$

or

$$m(m^2 + n^2 + o^2) + a(-m^2 + n^2 + o^2) = 0. \quad (4)$$

We see that the locus is an algebraic surface of third degree.

To display the locus of (4) in Maple it is advantageous to express it in a parametric form. Considering the intersection of a line $l := X = [t, ut, vt]$ through the singular point $O = [0, 0, 0]$ with the surface (4) we obtain $t = a(1 - u^2 - v^2)/(1 + u^2 + v^2)$ which implies the following parametric equations of the surface (4)

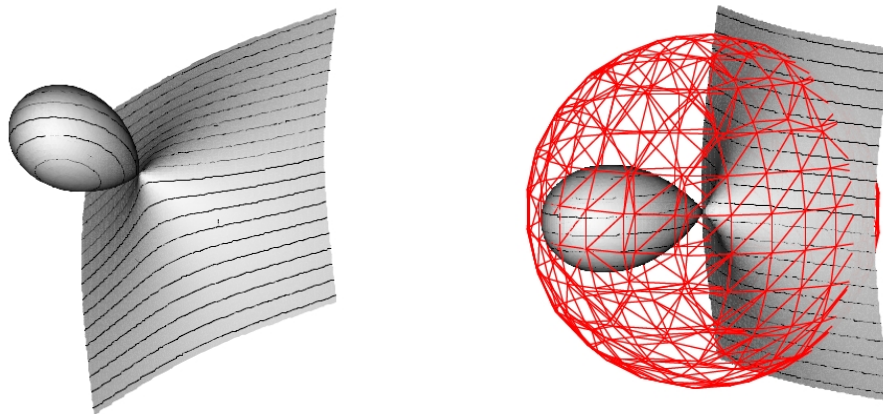


Figure 14: 3D version of a strophoid for $a = 1$ Figure 15: 3D version of a strophoid with sphere for $a = 1$

$$\begin{aligned} p &= \frac{a(1 - u^2 - v^2)}{1 + u^2 + v^2} \\ q &= \frac{au(1 - u^2 - v^2)}{1 + u^2 + v^2} \\ r &= \frac{av(1 - u^2 - v^2)}{1 + u^2 + v^2} \end{aligned} \quad (5)$$

for real parameters u, v . Fig. 14 displays a 3D strophoid and Fig. 15 displays 3D strophoid together with the sphere — in both cases the strophoid is expressed in a parametric form.

5 Conclusion

The development of computers and mathematical software allows such activities that previously were not possible. In the examples that were solved above we see how effective computer support actually is. New technologies show new possibilities for exploring loci not only in plane, but also in space.

Some open questions [13]:

- To what extent should we teach the theory of automated geometry theorem proving (Gröbner bases computations or Wu–Ritt method, normal forms, theory of elimination etc.)? (from “black box” to perfect understanding)
- Is it necessary to include automatic theorem proving into mathematics teachers’ curricula?
- Is it necessary to introduce automatic theorem proving (in reduced form) into secondary schools curricula (15-18 years of age)?

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